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Odd Triperfect Numbers Are Divisible By Eleven Distinct Prime Factors

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Abstract. We prove that an odd triperfect number has at least eleven distinct prime factors.

1. Introduction. A positive number N is called a triperfect number if $\sigma(N) = 3N$ where $\sigma(N)$ is the sum of the positive divisors of N. Six even triperfect numbers are known:

$$2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151,$$
$$2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127,$$
$$2^{9} \cdot 3 \cdot 11 \cdot 31,$$
$$2^{8} \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73,$$
$$2^{5} \cdot 3 \cdot 7,$$
$$2^{3} \cdot 3 \cdot 5.$$

However, the existence of an odd triperfect (OT) number is an open question. McDaniel [4] and Cohen [2] proved that an OT number has at least nine distinct prime factors; the author proved that it has at least ten prime factors [3], and Beck and Najar [1] showed that it exceeds 10^{50}.

In this paper we prove

THEOREM. *If N is OT, N has at least eleven distinct prime factors.*

2. Proof of Theorem. Throughout this paper we let

$$N = \prod_{i=1}^{10} p_i^{a_i},$$

where p_i's are odd primes, $p_1 < \cdots < p_{10}$ and a_i's are positive integers. We call $p_i^{a_i}$ a component of N and write $p_i^{a_i} \,||\, N$.

The following lemmas are easy to prove:

LEMMA 1. *If N is OT, a_i's are even for $1 \leqslant i \leqslant 10$.*

LEMMA 2. *If N is OT and q is a prime factor of $\sigma(p_i^{a_i})$ for some i, then $q = 3$ or $q = p_j$ for some j, $1 \leqslant j \leqslant 10$.*

The following lemmas are stated in [5].

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LEMMA 3. *Suppose q is a prime, $q \geqslant 2$ and $a \geqslant 1$. Then $\sigma(q^a)$ has a prime factor p such that $a+1$ is the order of q modulo p except for $q = 2$ and $a = 5$ and for $q =$ a Mersenne prime and $a = 1$. In particular $a + 1 \mid p - 1$.*

LEMMA 4. *Suppose p is a Fermat prime* (3, 5, 17, *etc.*), *q is an odd prime and a is even. If $p^b \mid \sigma(q^a)$, then $q \equiv 1$ (p), $p^b \mid a+1$, and $\sigma(q^a)$ has b distinct prime factors congruent to 1 modulo p.*

LEMMA 5. *If N is OT*, $17 \nmid N$.

Proof. Suppose N is OT. Since the three smallest primes $\equiv 1$ (17) are 103, 137, and 239 and

$$\frac{3}{2}\,\frac{5}{4}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{12}\,\frac{17}{16}\,\frac{19}{18}\,\frac{103}{102}\,\frac{137}{136}\,\frac{239}{238} < 3,$$

N has at most two primes $\equiv 1$ (17). Suppose p^a and q^b are components of N and $p \equiv q \equiv 1$ (17). If $17^c \mid N$ and $c \geqslant 4$, then $17^2 \mid \sigma(p^a)$ or $17^2 \mid \sigma(q^b)$, and, by Lemma 4, N would have two more primes $\equiv 1$ (17), a contradiction. Hence $17^4 \nmid N$. Suppose $17^2 \| N$. Then N has a component 307^d because $\sigma(17^2) = 307$. Then $17 \nmid \sigma(307^d)$ because $16661 \cdot 36857 \mid \sigma(307^{16})$, $\sigma(307^{16}) \mid \sigma(307^d)$ and

$$\frac{3}{2}\,\frac{5}{4}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{12}\,\frac{17}{16}\,\frac{19}{18}\,\frac{307}{306}\,\frac{16661}{16660}\,\frac{36857}{36856} < 3.$$

Hence N has another component p^b such that $17^2 \mid \sigma(p^b)$. Then we get a contradiction again. Hence $17 \nmid N$. Q.E.D.

The proof of the following lemma is easy.

LEMMA 6. *If N is OT, $p_9 \leqslant 283$.*

LEMMA 7. *If N is OT and $5^a \| N$, then $a = 2$, $5^2 \mid \sigma(P_{10}^{a_{10}})$ and $p_{10} \geqslant 311$.*

Proof. Suppose N is OT, p^b is a component of N and $5 \mid \sigma(p^b)$. By Lemma 4, $p \equiv 1$ (5), $5 \mid b+1$ and $\sigma(p^4) \mid \sigma(p^b)$. If $61 \leqslant p \leqslant 281$, then

$\sigma(p^4)$ has a prime factor q such that

$$\frac{3}{2}\,\frac{5}{4}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{12}\,\frac{19}{18}\,\frac{23}{22}\,\frac{29}{28}\,\frac{p}{p-1}\,\frac{q}{q-1} < 3, \quad \text{or}$$

$\sigma(p^4)$ has prime factors q and r such that

$$\frac{3}{2}\,\frac{5}{4}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{12}\,\frac{19}{18}\,\frac{23}{22}\,\frac{p}{p-1}\,\frac{q}{q-1}\,\frac{r}{r-1} < 3.$$

Hence $p = 11$, 31 or 41 or $p \geqslant 311$.

Suppose $p = 11$, 31 or 41. If $5^2 \mid \sigma(p^b)$, $5^2 \mid b+1$ by Lemma 4. Then $\sigma(p^{24}) \mid \sigma(p^b)$ and $\sigma(p^{24})$ has two distinct prime factors > 283, contradicting Lemma 6. Hence $5^2 \nmid \sigma(p^b)$. Since $3221 \mid \sigma(11^4)$, $17351 \mid \sigma(31^4)$ and $579281 \mid \sigma(41^4)$, $5^2 \nmid \sigma(\prod_{i=1}^{9} p_i^{a_i})$ and $p_{10} = 3221$, 17351 or 579281 and $5 \mid \sigma(p_{10}^{a_{10}})$. However, $\sigma(p_{10}^4)$ has a prime factor > 283, contradicting Lemma 6. Hence $p \geqslant 311$ and $5^a \mid \sigma(p^b)$.

If $a \geqslant 4$, then by Lemma 4, N would have four more primes $\equiv 1$ (5), which is a contradiction because

$$\frac{3}{2}\,\frac{5}{4}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{12}\,\frac{19}{18}\,\frac{31}{30}\,\frac{41}{40}\,\frac{61}{60}\,\frac{311}{310} < 3.$$

Hence $a = 2$. Q.E.D.

LEMMA 8. *If N is OT, $p_9 \leqslant 71$.*

Proof. By Lemma 6, $31 = \sigma(5^2) \mid N$. Since

$$\frac{3}{2}\,\frac{\sigma(5^2)}{5^2}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{12}\,\frac{19}{18}\,\frac{23}{22}\,\frac{31}{30}\,\frac{73}{72}\,\frac{311}{310} < 3,$$

$p_9 \leqslant 71$. Q.E.D.

Proof of Theorem. If N is OT, then by Lemmas 4 and 7, $5^2 \mid \sigma(p_{10}^{a_{10}})$, $5^2 \mid a_{10} + 1$ and $\sigma(p_{10}^{24}) \mid (p_{10}^{a_{10}})$. By Lemma 3, $\sigma(p_{10}^{24})$ has a prime factor q such that $25 \mid q - 1$. Hence $q = 25b + 1$ for some b. Since q is a prime, $b \neq 1$ or 2. Then $q > 71$ and $q \neq p_{10}$, contradicting Lemma 8. Q.E.D.

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